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On the Simplicity of Haag Fields†

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Abstract

It is shown that the simplicity of the C^*-algebra of quasilocal observables of a Haag field can be deduced from a postulate which seems to be plausible from a physical point of view.

1. *Introduction*

A Haag field (Misra, 1965) $\langle H, \bar{Q}, O \to R(O)\rangle$ is defined as follows: To every bounded open domain O in Minkowski space we associate a von Neumann algebra (Dixmier, 1957) $R(O)$ in a separable Hilbert space H. $R(O)$ is generated by the observables (of a given physical system) which can be measured in the domain O.

The algebras $R(O)$ are called *algebras of local observables*. The union of all these algebras

$$Q = \bigcup_{O} R(O)$$

is a *-algebra, the *algebra of all local observables*. If we complete Q with respect to the norm topology we get the C^*-algebra $\bar{Q}$ of *quasilocal observables*.

The structure $\langle H, \bar{Q}, O \to R(O)\rangle$ is called a *Haag field*.

In a purely algebraic formulation of the quantum theory of fields (Haag & Kastler, 1964) the algebras $R(O)$ are considered as abstract C^*-algebras instead of von Neumann algebras. From the mathematical point of view it is, however, advantageous to define these algebras as sets of operators acting in a Hilbert space.

Haag & Kastler (1964) have shown that two representations of the algebra $\bar{Q}$ of quasilocal observables are physically equivalent (i.e. contain the same physical information) if they have the same kernel.

A stronger definition of physical equivalence was given by Misra (1965): Two representations $D_1(\bar{Q})$ and $D_2(\bar{Q})$ are physically equivalent in the sense of Misra if they are *-isomorphic and locally unitarily equivalent (i.e. $D_1[R(O)]$ and $D_2[R(O)]$ are unitarily equivalent for every bounded open domain O).

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Misra (1965) showed that all faithful representations of $\bar{Q}$ are physically equivalent in this sense if the local algebras $R(O)$ are infinite, and it is well known that the algebras $R(O)$ are indeed of infinite type (Guenin & Misra, 1963; Kadison, 1963; Borchers, 1967).

Hence, if $\bar{Q}$ is simple, all its non-trivial representations are physically equivalent in the sense of Haag and Kastler as well as in the sense of Misra. It is therefore interesting to know whether $\bar{Q}$ is simple. Misra (1965) has shown that this is the case if the Haag field $\langle H, \bar{Q}, O \to R(O)\rangle$ has a certain property which he calls 'property F'. Borchers (1967) proved that $\bar{Q}$ is simple if the centre of Q is trivial.

It is the purpose of this paper to show that the centre of Q is trivial and that, therefore, the C^*-algebra $\bar{Q}$ of quasilocal observables is simple, provided the Haag field $\langle H, \bar{Q}, O \to R(O)\rangle$ satisfies the following postulate which seems to be plausible from a physical point of view. This postulate, which we refer to as *extended locality*, states that if O_1 and O_2 are totally space-like to each other then the intersection of $R(O_1)$ and $R(O_2)$ is trivial.

In the last section we deduce the property F from the fact that the centre of Q is trivial. To do this we need, however, rather strong assumptions which are not needed in Borchers's proof (Borchers, 1967).

2. *The Postulates*

We denote by O a (bounded) open region in Minkowski space and by O' the region which is totally space-like relative to O.

If M is a set of (bounded) operators in a separable Hilbert space H, we denote by M' the commutant and by M'' the bicommutant of M. M'' is the von Neumann algebra generated by M.

Let $O \to R(O)$ be a correspondence between open space-time domains O and the von Neumann algebras $R(O)$ of local observables. We state the following postulates:

2.1. *Isotony*

$$O_1 \subset O_2 \Rightarrow R(O_1) \subset R(O_2)$$

2.2. *Translation Covariance*

The Hilbert space H in which the algebras $R(O)$ are defined is the representation space of a unitary representation of the translation group:

$$U(x) = \int \exp(ipx)\, dE(p)$$

and we have

$$U(x)\, R(O)\, U^{-1}(x) = R(O + x)$$

2.3. *Spectrum Condition*

Let $\overline{V}_+$ be the closed forward cone and $E(\Delta)$ the spectral measure of the translation operator U for a set Δ in energy-momentum space. Then

$$\Delta \cap \overline{V}_+ = \Phi \Rightarrow E[\triangle] = 0$$

2.4. *Weak Additivity*

Let O be an arbitrary bounded open space-time domain. Then

$$Q'' = \left\{\bigcup_x R(O+x)\right\}''$$

2.5. *Locality*

$$O_1 \subset O_2' \Rightarrow R(O_1) \subset R(O_2)'$$

2.6. *Extended Locality*

$$O_1 \subset O_2' \Rightarrow R(O_1) \cap R(O_2) = \{\lambda . 1\} \qquad (2.6.1)$$

If (2.6.1) is not true then there exists an observable A which belongs simultaneously to $R(O_1)$ and to $R(O_2)$. If we measure A in O_1 we know at the same time its value in O_2 which contradicts physical intuition if the two regions are totally space-like to each other.†

We remark that there is a connection between extended locality and strict locality: The former can be deduced from the latter. Strict locality is defined as follows (Licht, 1963; Kraus, 1964):

Let O_1 and O_2 be totally space-like to each other. Then for every non-trivial projector $P \in R(O_1)$ and for every vector $\phi \in H$ there exists a vector $\psi \in PH$ such that

$$(\phi, A\phi) = (\psi, A\psi) \, [\forall \, A \in R(O_2)]$$

Kraus (1964) deduced strict locality from some other postulates under the additional assumption that Q'' is a factor.

The postulates 2.1–2.6 are sufficient to show that $\bar{Q}$ is simple. For our discussion of property F we need, however, some more assumptions.

A stronger form of weak additivity is

† The unit operator 1 plays a special role. Since by definition every von Neumann algebra contains a unit element we cannot have the empty set on the right-hand side of (2.6.1). We can interpret the unit operator as follows: Since 1 is a projector it belongs to a yes–no experiment. This experiment answers the question: 'Does our physical system exist?' Of course, the existence of this system is the most fundamental of our assumptions.

2.7. *Additivity*

$$R(O_1 \cup O_2) = \{R(O_1) \cup R(O_2)\}''$$

This amounts to the statement that the observables which can be measured in $O_1 \cup O_2$ are those which can be measured in O_1 plus those which can be measured in O_2, and no more.

Next we can write the postulate of isotony in the following way:

$$O_1 \subset O_2 \Rightarrow R(O_1 \cap O_2) = R(O_1) \cap R(O_2)$$

We sharpen this statement by requiring

2.8. *Continuous Isotony* (Kraus, 1964)

Let $O_1 \supset O_2 \supset \cdots \rightarrow \bigcap_i O_i$ be a monotone decreasing sequence of open sets in Minkowski space, and let $\bigcap_i O_i$ have a non-empty open interior O with $\bar{O} \supset \bigcap_i O_i$ ($\bar{O}$ is the closure of O). Then

$$R(O) = \bigcap_i R(O_i)$$

2.9. *Primitive Causality*

Let T be an open region in Minkowski space, containing a complete space-like hypersurface. Then

$$Q'' = \left\{\bigcup_{O \subset T} R(O)\right\}''$$

Primitive causality says that measurements in a finite time interval are sufficient to determine the behaviour of a physical system for all times. This is always the case if the system is described by a conventional field theory based on local fields satisfying hyperbolic equations of motion, because in this case the boundary values on a space-like hypersurface are sufficient to determine the evaluation of the system.

3. *The Centre of Q*

From postulates 2.1–2.5 Borchers (1967) proved

Theorem 3.1

Let $J \subset \bar{Q}$ be a norm-closed two-sided ideal in $\bar{Q}$. Then

$$J \neq \{0\} \Rightarrow J \cap Q \cap Q' \neq \{0\}$$

We want to show now that the centre $Q \cap Q'$ of Q is trivial. To this aim we need

Lemma 3.1

The algebra Q of all local observables does not contain non-trivial translation invariant elements.

Proof: Let $A \in Q$ be translation invariant. Then we have $A \in R(O)$ for some O. But because A is translation invariant we also have $A \in R(O+x)$ for arbitrarily large space-like x, in contradiction to Postulate 2.6.

From the spectrum condition Araki (1964) deduced

Lemma 3.2

Every element of the centre $Q' \cap Q''$ of Q'' commutes with all translations $U(x)$.

From these lemmas follows

Theorem 3.2

The centre $Q' \cap Q$ of the algebra Q of all local observables is trivial.

Proof: Because $Q \subset Q''$ we have

$$Q' \cap Q \subset Q' \cap Q''$$

Thus the elements of $Q' \cap Q$ are translation invariant (Lemma 3.2) and we therefore have $Q' \cap Q = \{\lambda . 1\}$ (Lemma 3.1).

Together with Theorem 3.1, this amounts to

Theorem 3.3

The postulates 2.1–2.6 imply that the algebra $\bar{Q}$ of quasilocal observables is simple.

4. *Property F*

A Haag field $\langle H, \bar{Q}, O \rightarrow R(O) \rangle$ is said to have the *property F* if for every bounded open domain O_1 there exists a bounded open domain $O_2 \supset O_1$ such that $R(O_2)$ is a factor.

Theorem 4.1 (Misra, 1965)

If a Haag field $\langle H, \bar{Q}, O \rightarrow R(O) \rangle$ has the property F, then the algebra Q of all local observables and the algebra $\bar{Q}$ of quasilocal observables are simple.

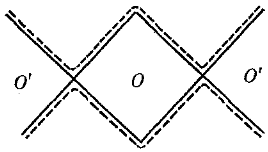
Now we make use of the postulates 2.7–2.9 to give a simple proof that the property F and thus the simplicity of $\bar{Q}$ follows from the fact that the centre of Q is trivial. Borchers's proof is, however, more elegant, because it works without the additional postulates 2.7–2.9 (Borchers, 1967).

Lemma 4.1

Let O be a bounded open domain such that $O = O''$. Then $R(O)$ is a factor.

Proof: We consider a monotone decreasing sequence of open regions T_i, for which

$$\lim_{i\to\infty} T_i = \bigcap_i T_i = O \cup O'$$



[This trick is due to Kraus (1964).] Because of continuous isotony (2.8) and primitive causality (2.9) we then have

$$R(O \cup O') = \bigcap_i R(O_i) = Q'' \cap Q'' \cdots \cap = Q''$$

or, making use of additivity (2.7):

$$Q'' = \{R(O) \cup R(O')\}''$$

The commutant of Q'' is

$$Q' = R(O)' \cap R(O')'$$

But because of locality (V) we have

$$R(O')' \supset R(O)$$

Thus

$$Q' \supset R(O)' \cap R(O)$$

and since $Q \supset R(O)$:

$$Q' \cap Q \supset R(O)' \cap R(O)$$

Hence if $R(O)$ is not a factor then the centre of Q is not trivial, in contradiction to Theorem 3.2.

From Lemma 4.1 follows immediately that the Haag field $\langle H, \bar{Q}, O \to R(O)\rangle$ has the property F.

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Note Added in Proof

The present form of postulate 2.8 is due to K. Kraus (private communication). I am indebted to Dr. Kraus for pointing out an inaccuracy in a first form of this postulate.